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Orthogonal Polynomials and Their Derivatives, I

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Complete characterization is given for all orthogonal polynomials whose derivatives are linear combinations of at most two polynomials of the same system.

Ever since 1915 when Luzin [16, p. 50] asked whether there are any orthogonal systems in addition to the trigonometric system that are invariant under either differentiation or integration there have been several investigations conducted towards finding all the orthogonal polynomials whose derivatives satisfy certain conditions. Such problems have been solved, for example, in [2, 4–13, 15, 19, 20].

In this paper we give a complete characterization of all orthogonal polynomials whose derivatives are linear combinations of at most two polynomials of the same system.

Let $d\alpha$ be a finite positive measure on the real line with infinite support and finite moments. Such a measure $d\alpha$ will be called a distribution and the corresponding system of orthonormal polynomials is denoted by $\{p_n\}_{n=0}^{\infty}$, where $p_n(x) = p_n(d\alpha, x) = \gamma_n(d\alpha)\, x^n + \cdots$, $\gamma_n > 0$. These polynomials p_n satisfy the three-term recurrence relation

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{1}$$

$n = 0, 1,\ldots$, where $a_0 = 0$, $a_n = \gamma_{n-1}/\gamma_n$, $n = 1, 2,\ldots$ and

$$b_n = \int_{-\infty}^{\infty} xp_n^2(x)\, d\alpha(x).$$

Our results are summarized in the following proposition.

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THEOREM. *Let* $\{p_n\}_{n=0}^{\infty}$ *be a system of orthonormal polynomials corresponding to some distribution* $d\alpha$. *Then the following statements are equivalent.*

(i) *There exist two integers* j *and* k *and two sequences* $\{e_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *such that* $j < k$ *and*

$$p_n' = e_n p_{n-j} + c_n p_{n-k}$$

for $n = 1, 2, \ldots$.

(ii) *There exists a nonnegative constant* c *such that*

$$p_n' = (n/a_n)\, p_{n-1} + c a_n a_{n-1} a_{n-2} p_{n-3}$$

for $n = 1, 2, \ldots$, *where* a_n *denotes the recursion coefficient in* (1).

(iii) *There exist three real numbers* c, b *and* K *such that* $c \geqslant 0$, *if* $c = 0$ *then* $K > 0$, *and the recursion coefficients* a_n *and* b_n *in* (1) *satisfy*

$$n = c a_n^2 [a_{n+1}^2 + a_n^2 + a_{n-1}^2] + K a_n^2$$

for $n = 1, 2, \ldots$ *and*

$$b_n = b$$

for $n = 0, 1, 2, \ldots$.

(iv) *The distribution* $d\alpha$ *is absolutely continuous and there exist four real numbers* D, c, b *and* K *such that* $D > 0$, $c \geqslant 0$, *if* $c = 0$ *then* $K > 0$, *and*

$$\alpha'(x) = D \exp\left[-\frac{c}{4}(x-b)^4 - \frac{K}{2}(x-b)^2\right]$$

for $-\infty < x < \infty$.

Moreover, if c *is given by one of the statements* (ii), (iii) *or* (iv) *then in the remaining statements it has the same value. The same comment applies to* b *and* K *in* (iii) *and* (iv). *If* c *is given by* (ii) *then* b *and* K *in* (iii) *and* (iv) *would still be arbitrary except if* $c = 0$ *then* K *must be positive.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (ii). We need to show (iii) $\Leftarrow$ (iv) because of the comments made about c, b and K.

(i) $\Rightarrow$ (ii): Since p_n' is a polynomial of degree $n-1$ the index j must be 1 and by comparing leading coefficients we obtain $e_n = n/a_n$. Hence

$$p_n' = (n/a_n)\, p_{n-1} + c_n p_{n-k}. \tag{2}$$

First assume that $k = 2$. Then

$$p_n' = (n/a_n)\, p_{n-1} + c_n p_{n-2}. \tag{3}$$

Differentiating the recurrence formula (1) and evaluating p_{n+1}', p_n' and p_{n-1}' by (3) we obtain

$$\begin{aligned} x \left(\frac{n}{a_n} p_{n-1} + c_n p_{n-2}\right) &+ p_n \\ &= (n+1)\, p_n + \left(a_{n+1} c_{n+1} + b_n \frac{n}{a_n}\right) p_{n-1} \\ &\quad + \left(b_n c_n + a_n \frac{n-1}{a_{n-1}}\right) p_{n-2} + a_n c_{n-1} p_{n-3}. \end{aligned}$$

Expressing here xp_{n-2} in terms of the recurrence formula and dividing both sides by n/a_n we get

$$\begin{aligned} xp_{n-1} = a_n p_n &+ \left(\frac{a_n a_{n+1} c_{n+1}}{n} + b_n - \frac{a_{n-1} a_n c_n}{n}\right) p_{n-1} \\ &+ \left(\frac{a_n b_n c_n}{n} + \frac{a_n^2 (n-1)}{a_{n-1} n} - \frac{a_n b_{n-2} c_n}{n}\right) p_{n-2} \\ &+ \left(\frac{a_n^2 c_{n-1}}{n} - \frac{a_n a_{n-2} c_n}{n}\right) p_{n-3}, \end{aligned}$$

which compared with the recurrence formula leads to

$$a_n c_{n-1} = a_{n-2} c_n, \tag{4}$$

$$n a_{n-1}^2 = a_n a_{n-1} b_n c_n - a_n a_{n-1} b_{n-2} c_n + (n-1)\, a_n^2 \tag{5}$$

and

$$b_{n-1} = b_n + \frac{a_n a_{n+1} c_{n+1}}{n} - \frac{a_{n-1} a_n c_n}{n}. \tag{6}$$

It follows from (4) that

$$\frac{c_{n-1}}{a_{n-1} a_{n-2}} = \frac{c_n}{a_n a_{n-1}}, \qquad n = 3, 4, \ldots$$

so that there exists a constant c such that

$$c_n = c a_n a_{n-1}, \qquad n = 2, 3, \ldots. \tag{7}$$

Substituting (7) into (5) and (6) we obtain

$$na_{n-1}^2 = ca_n^2 a_{n-1}^2(b_n - b_{n-2}) + (n-1)\, a_n^2 \tag{8}$$

and

$$b_{n-1} = b_n + \frac{ca_n^2}{n}\,[a_{n+1}^2 - a_{n-1}^2] \tag{9}$$

for $n = 3, 4, \ldots$. Now we can use (9) to evaluate $b_n - b_{n-2}$ in (8) and proceeding this way we get

$$\begin{aligned} na_{n-1}^2 = {} & c^2 a_n^2 a_{n-1}^4 (n-1)^{-1}(a_{n-2}^2 - a_n^2) \\ & + c^2 a_n^4 a_{n-1}^2 n^{-1}(a_{n-1}^2 - a_{n+1}^2) + (n-1)\, a_n^2, \end{aligned}$$

which we can rewrite as

$$\frac{n}{a_n^2} - \frac{n-1}{a_{n-1}^2} = \frac{c^2 a_{n-1}^2 a_{n-2}^2}{n-1} - \frac{c^2 a_{n+1}^2 a_n^2}{n+1} - \frac{c^2 a_{n+1}^2 a_n^2}{n(n+1)} - \frac{c^2 a_n^2 a_{n-1}^2}{(n-1)n}. \tag{10}$$

Since $a_n^2 > 0$ for $n = 1, 2, \ldots$ we obtain from (10) that the sequence

$$\frac{n}{a_n^2} + \frac{c^2 a_{n+1}^2 a_n^2}{n+1} + \frac{c^2 a_n^2 a_{n-1}^2}{n}$$

decreases for $n = 3, 4, \ldots$. Therefore there exists a constant $A > 0$ such that

$$\frac{n}{a_n^2} + \frac{c^2 a_{n+1}^2 a_n^2}{n+1} + \frac{c^2 a_n^2 a_{n-1}^2}{n} \leqslant A, \qquad n = 3, 4, \ldots$$

and thus

$$n \leqslant A a_n^2 \tag{11}$$

and

$$c^2 a_{n+1}^2 a_n^2 \leqslant A(n+1) \tag{12}$$

for $n = 3, 4, \ldots$. From (11) and (12) we conclude that

$$c^2 \leqslant \frac{A(n+1)}{a_{n+1}^2 a_n^2} \leqslant \frac{A^3}{n}$$

and letting $n \to \infty$ we get $c = 0$ so that by (7) formula (3) takes the form

$$p_n' = (n/a_n)\, p_{n-1}, \qquad n = 2, 3, \ldots$$

which proves (ii) when $k=2$ with $c=0$. Next let $k=3$. Then we have to show that c_n in (2) satisfies

$$c_n = ca_n a_{n-1} a_{n-2}, \qquad n=3, 4,... \tag{13}$$

with some constant $c \geqslant 0$. We have

$$p'_n = (n/a_n)\, p_{n-1} + c_n p_{n-3}. \tag{14}$$

First we will derive some relationships which we will use in establishing (ii) $\Rightarrow$ (iii) as well. If we differentiate the recurrence formula (1) and substitute p'_{n+1}, p'_n and p'_{n-1} by the expression obtained from (14) then we get

$$\begin{aligned} x &\left(\frac{n}{a_n} p_{n-1} + c_n p_{n-3}\right) + p_n \\ &= (n+1)\, p_n + n\frac{b_n}{a_n} p_{n-1} \\ &\quad + \left(a_{n+1} c_{n+1} + a_n \frac{n-1}{a_{n-1}}\right) p_{n-2} + b_n c_n p_{n-3} + a_n c_{n-1} p_{n-4} \end{aligned}$$

and applying the recurrence formula to xp_{n-1} and xp_{n-3} we end up with

$$\begin{aligned} np_n &+ \frac{n}{a_n} b_{n-1} p_{n-1} + \left(\frac{n}{a_n} a_{n-1} + c_n a_{n-2}\right) p_{n-2} + c_n b_{n-3} p_{n-3} \\ &+ c_n a_{n-3} p_{n-4} + p_n = (n+1)\, p_n + n\frac{b_n}{a_n} p_{n-1} \\ &+ \left(a_{n+1} c_{n+1} + a_n \frac{n-1}{a_{n-1}}\right) p_{n-2} + b_n c_n p_{n-3} + a_n c_{n-1} p_{n-4}. \end{aligned} \tag{15}$$

Comparing the coefficients in (15) we obtain

$$b_n = b_{n-1}, \tag{16}$$

$$\frac{n}{a_n} a_{n-1} + c_n a_{n-2} = \frac{n-1}{a_{n-1}} a_n + c_{n+1} a_{n+1}, \tag{17}$$

$$c_n b_{n-3} = c_n b_n \tag{18}$$

and

$$c_n a_{n-3} = c_{n-1} a_n \tag{19}$$

for $n=2, 3,...$. Now (13) follows from (19) with some constant c. In order to show that c in (13) is nonnegative we apply $c_n = ca_n a_{n-1} a_{n-2}$ to (17) and we obtain

$$\frac{n}{a_n} a_{n-1} + ca_n a_{n-1} a_{n-1}^2 = \frac{n-1}{a_{n-1}} a_n + ca_{n+1}^2 a_n a_{n-1},$$

which we rewrite in the form

$$\frac{n}{a_n^2} - c(a_{n+1}^2 + a_n^2 + a_{n-1}^2) = \frac{n-1}{a_{n-1}^2} - c(a_n^2 + a_{n-1}^2 + a_{n-2}^2)$$

for $n = 2, 3, \ldots$. Hence there exists a constant K such that

$$(n/a_n^2) - c(a_{n+1}^2 + a_n^2 + a_{n-1}^2) = K \tag{20}$$

for $n = 1, 2, \ldots$. If c is negative then $K > 0$ and $n/a_n^2 < K$, $-ca_n^2 < K$ so that $-nc < k^2$, $n = 1, 2, \ldots$, which is impossible. Thus c in (13) is nonnegative. Consequently we have proved (i) $\Rightarrow$ (ii) when k in (2) equals 3. Now let $k > 3$ in (2). Differentiating again the recurrence formula (1) and applying (2) we obtain

$$\begin{aligned} x\left(\frac{n}{a_n} p_{n-1} + c_n p_{n-k}\right) + p_n \\ &= (n+1) p_n + n\frac{b_n}{a_n} p_{n-1} + a_n \frac{n-1}{a_{n-1}} p_{n-2} \\ &\quad + a_{n+1} c_{n+1} p_{n-k+1} + b_n c_n p_{n-k} + a_n c_{n-1} p_{n-k-1} \end{aligned}$$

so that by the recurrence formula (1)

$$\begin{aligned} np_n + \frac{n}{a_n} b_{n-1} p_{n-1} + \frac{n}{a_n} a_{n-1} p_{n-2} + c_n a_{n-k+1} p_{n-k+1} \\ + c_n b_{n-k} p_{n-k} + c_n a_{n-k} p_{n-k-1} + p_n \\ = (n+1) p_n + n\frac{b_n}{a_n} p_{n-1} + a_n \frac{n-1}{a_{n-1}} p_{n-2} + a_{n+1} c_{n+1} p_{n-k+1} \\ + b_n c_n p_{n-k} + a_n c_{n-1} p_{n-k-1}. \end{aligned}$$

Hence

$$\begin{aligned} b_n &= b_{n-1}, \\ \frac{n}{a_n} a_{n-1} &= \frac{n-1}{a_{n-1}} a_n, \end{aligned} \tag{21}$$

$$\begin{aligned} c_n a_{n-k+1} &= a_{n+1} c_{n+1}, \\ c_n b_{n-k} &= b_n c_n \end{aligned} \tag{22}$$

and

$$c_n a_{n-k} = c_{n-1} a_n \tag{23}$$

for $n = 2, 3, 4, \ldots$. By (23), $c_{n+1} a_{n-k+1} = c_n a_{n+1}$ and thus by (22)

$$c_n c_{n+1} a_{n-k+1}^2 = c_n c_{n+1} a_{n+1}^2. \tag{24}$$

It follows from (21) that

$$\frac{a_{n+1}^2}{n+1} = \frac{a_{n-k+1}^2}{n-k+1} \tag{25}$$

for $n = k, k+1, \ldots$ and substituting (25) into (24) we obtain

$$c_n c_{n+1} = c_n c_{n+1} \frac{n+1}{n-k+1}$$

for $n = k, k+1, \ldots$. Hence $c_n c_{n+1} = 0$ for $n = k, k+1, \ldots$ and by (22) $c_n = 0$ for $n = k, k+1, \ldots$. Thus again we see that (ii) holds with $c = 0$.

(ii) $\Rightarrow$ (iii): We proved that (14) implies (16) and (2) which is equivalent to (iii) provided that $b_1 = b_0$ as well. We can show $b_1 = b_0$ as follows. We have

$$xp_1 = a_2 p_2 + b_1 p_1 + a_1 p_0$$

and differentiating this and using

$$p_2' = (2/a_2)\, p_1$$

we obtain

$$x\gamma_1 + p_1 = 2p_1 + b_1 \gamma_1$$

so that

$$p_1 = \gamma_1 (x - b_1). \tag{26}$$

By the recurrence formula

$$xp_0 = a_1 p_1 + b_0 p_0.$$

Hence

$$p_1 = \gamma_1(x - b_0). \tag{27}$$

Comparing (26) and (27) we can conclude that $b_1 = b_0$.

(iv) $\Rightarrow$ (ii): Let d be given by (iv). Then integration by parts yields

$$\begin{aligned} \int_{-\infty}^{\infty} p_n'(x)\, p_l(x)\, d\alpha(x) &= \int_{-\infty}^{\infty} [p_n(x)\, p_l(x)]'\, d\alpha(x) \\ &= \int_{-\infty}^{\infty} p_n(x)\, p_l(x)[c(x-b)^3 + K(x-b)]\, d\alpha(x) \end{aligned} \tag{28}$$

for $n > l \geqslant 0$. Hence

$$\int_{-\infty}^{\infty} p_n'(x)\, p_l(x)\, d\alpha(x) = 0, \qquad 0 \leqslant l < n-3 \tag{29}$$

and since $d\alpha$ is symmetric with respect to b, $p_n(x)\, p_{n-2}(x)$ is an even polynomial in the variable $(x-b)$ so that by (28)

$$\int_{-\infty}^{\infty} p_n'(x)\, p_{n-2}(x)\, d\alpha(x) = 0. \tag{30}$$

Moreover,

$$\begin{aligned} \int_{-\infty}^{\infty} p_n'(x)\, p_{n-1}(x)\, d\alpha(x) &= \int_{-\infty}^{\infty} [n\gamma_n x^{n-1} + \cdots]\, p_{n-1}(x)\, d\alpha(x) \\ &= \frac{n}{a_n} \int_{-\infty}^{\infty} [\gamma_{n-1} x^{n-1} + \cdots]\, p_{n-1}(x)\, d\alpha(x) = \frac{n}{a_n} \end{aligned} \tag{31}$$

and by (28)

$$\begin{aligned} \int_{-\infty}^{\infty} p_n'(x)\, p_{n-3}(x)\, d\alpha(x) &= c \int_{-\infty}^{\infty} p_n(x)\, p_{n-3}(x)(x-b)^3\, d\alpha(x) \\ &= c \int_{-\infty}^{\infty} p_n(x)[\gamma_{n-3} x^n + \cdots]\, d\alpha(x) \\ &= c\,\frac{\gamma_{n-3}}{\gamma_n} \int_{-\infty}^{\infty} p_n(x)[\gamma_n x^n + \cdots]\, d\alpha(x) \\ &= c\,\frac{\gamma_{n-3}}{\gamma_n} = c\,\frac{\gamma_{n-1}}{\gamma_n}\,\frac{\gamma_{n-2}}{\gamma_{n-1}}\,\frac{\gamma_{n-3}}{\gamma_{n-2}} = c a_n a_{n-1} a_{n-2}. \end{aligned} \tag{32}$$

It follows from (29), (30), (31) and (32) that the Fourier series expansion of p_n' in the system $\{p_l\}$ is given by

$$p_n' = (n/a_n)\, p_{n-1} + c a_n a_{n-1} a_{n-2} p_{n-3},$$

which establishes (ii).

(iv) $\Rightarrow$ (iii): If $d\alpha$ is defined by (iv) then it is symmetric around b so that all the coefficients b_n in the recurrence formula (1) equal b. We can find the coefficients a_n from (28), (31) and the recurrence formula. Since $b_n = b$ for $n = 0, 1, 2,\ldots$, we have

$$\begin{aligned}(x-b)\, p_{n-1} &= a_n p_n + a_{n-1} p_{n-2},\\ (x-b)^2 p_{n-1} &= a_n a_{n+1} p_{n+1} + (a_n^2 + a_{n-1}^2)\, p_{n-1} + a_{n-1} a_{n-2} p_{n-3}\end{aligned} \tag{33}$$

and

$$\begin{aligned}(x-b)^3 p_{n-1} = {} & a_n a_{n+1} a_{n+2} p_{n+2} + a_n(a_{n+1}^2 + a_n^2 + a_{n-1}^2)\, p_n \\ & + a_{n-1}(a_n^2 + a_{n-1}^2 + a_{n-2}^2)\, p_{n-2} + a_{n-1} a_{n-2} a_{n-3} p_{n-4}.\end{aligned} \tag{34}$$

Combining (28), (31), (33) and (34) we obtain

$$n/a_n = c a_n (a_{n+1}^2 + a_n^2 + a_{n-1}^2) + K a_n$$

and thus (iii) holds.

(iii) $\Rightarrow$ (iv): The moment problem for $d\alpha$ in (iv) has a unique solution [3, p. 80]. Hence it suffices to show that for any given real c, b and K such that $c \geqslant 0$ and $K > 0$ if $c = 0$, the equations

$$n = c a_n^2 [a_{n+1}^2 + a_n^2 + a_{n-1}^2] + K a_n^2, \tag{35}$$

$n = 1, 2,\ldots, a_0^2 = 0$ and

$$b_n = b,$$

$n = 0, 1, 2,\ldots$, have unique real solutions $\{a_n\}$ and $\{b_n\}$ such that $a_n > 0$ for $n = 1, 2,\ldots$. Obviously it is sufficient to examine solutions of (35). Moreover, if $c = 0$ in (35) then a_n^2 is uniquely determined. Now suppose that c in (35) is positive. When $K = 0$ in (35) then the uniqueness of $\{a_n\}$ was proved in [14] and [17]. For arbitrary K we use the following argument. Let $\{a_n\}$ be a sequence satisfying (35) and assume that $a_n > 0$ for $n = 1, 2,\ldots$. Define the sequence of polynomials $\{q_n\}$ $(n = 0, 1,\ldots)$ by the recurrence formula

$$x q_n(x) = a_{n+1} q_{n+1}(x) + a_n q_{n-1}(x), \tag{36}$$

$n = 0, 1,\ldots, q_0(x) = 1$. By Favard's theorem [3, p. 60] there exists a distribution $d\beta$ such that the polynomials q_n are orthonormal with respect to $d\beta$. We have by (35)

$$n/a_n > ca_n^3 + Ka_n$$

from which

$$a_n^2 \leqslant \frac{-K + \sqrt{K^2 + 4nc}}{2c}$$

so that

$$\sum_{n=1}^{\infty} a_n^{-1} = \infty.$$

Thus by Karleman's theorem [21, p. 59] the distribution $d\beta$ is uniquely determined by the sequence $\{a_n\}$ and by q_0. It follows from (36) that q_n is either even or odd depending whether n is even or odd. Hence

$$\int_{-\infty}^{\infty} x^{2n+1}\, d\beta(x) = 0, \qquad n = 0, 1, 2,\ldots . \tag{37}$$

Moreover, we have

$$\int_{-\infty}^{\infty} xq_n(x)\, q_{n-1}(x)\, d\beta(x) = a_n,$$

$$\int_{-\infty}^{\infty} x^3 q_n(x)\, q_{n-1}(x)\, d\beta(x) = a_n(a_{n+1}^2 + a_n^2 + a_{n-1}^2)$$

and

$$\int_{-\infty}^{\infty} [q_n(x)\, q_{n-1}(x)]'\, d\beta(x) = \frac{n}{a_n}$$

for $n = 1, 2,\ldots,$ so that by (35)

$$\int_{-\infty}^{\infty} [q_n(x)\, q_{n-1}(x)]'\, d\beta(x) = \int_{-\infty}^{\infty} q_n(x)\, q_{n-1}(x)[cx^3 + Kx]\, d\beta(x)$$

for $n = 1, 2,\ldots$. Since $q_n q_{n-1}$ is an odd polynomial of degree exactly $2n-1$, the system $\{q_n q_{n-1}\}$ spans all odd polynomials. Thus

$$(2n-1)\int_{-\infty}^{\infty} x^{2n-2}\, d\beta(x) = \int_{-\infty}^{\infty} x^{2n-1}[cx^3 + Kx]\, d\beta(x). \tag{38}$$

We obtain from (37) and (38) that

$$(n+1)\int_{-\infty}^{\infty} x^n \, d\beta(x) = \int_{-\infty}^{\infty} x^{n+1}[cx^3 + Kx] \, d\beta(x) \tag{39}$$

for $n = 0, 1, 2, \ldots$. We can rewrite (39) in terms of the moments μ_n of $d\beta$ as

$$(n+1)\,\mu_n = c\mu_{n+4} + K\mu_{n+2}, \qquad n = 0, 1, 2, \ldots . \tag{40}$$

Now we will show that

$$\lim_{n\to\infty} n^{-1}(\mu_n)^{1/n} = 0. \tag{41}$$

Let n be even. Then by (40)

$$c\mu_{n+4} \leqslant (n + 1 + |K|) \max\{\mu_{n+2}, \mu_n\}$$

and applying this inequality repeatedly we obtain

$$c^{N/2}\mu_N \leqslant (N + |k| - 3)^{(N-2)/2} \max\{\mu_2, \mu_0\}$$

for $N = 4, 6, \ldots$, from which (41) follows since by (37) $\mu_n = 0$ for n odd. It follows from (39) and (41) that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (n+1) \int_{-\infty}^{\infty} x^n \, d\beta(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} x^n [cx^4 + Kx^2] \, d\beta(x)$$

is an entire function of t, and interchanging summation and integration we obtain

$$\int_{-\infty}^{\infty} (tx+1)\, e^{tx} \, d\beta(x) = \int_{-\infty}^{\infty} e^{tx}[cx^4 + Kx^2] \, d\beta(x). \tag{42}$$

Since by (37)

$$\int_{-\infty}^{\infty} [cx^3 + Kx] \, d\beta(x) = 0, \tag{43}$$

integration of (42) with respect to t yields

$$t\int_{-\infty}^{\infty} e^{tx} \, d\beta(x) = \int_{-\infty}^{\infty} e^{tx}[cx^3 + Kx] \, d\beta(x). \tag{44}$$

Letting $t = iu$, u real, and integrating the right side of (44) by parts we obtain by (43)

$$\int_{-\infty}^{\infty} e^{iux}\, d\beta(x) = -\int_{-\infty}^{\infty} e^{iux} \int_{-\infty}^{x} [cy^3 + Ky]\, d\beta(y)\, dx. \tag{45}$$

It follows from (44) that

$$\lim_{u \to \infty} \int_{-\infty}^{\infty} e^{iux}\, d\beta(x) = 0.$$

Thus by Wiener's theorem [22, p. 261]

$$\beta(x) = \int_{-\infty}^{x} d\beta(y)$$

is a continuous function of x. Now we can apply the inverse Fourier transformation to both sides of (45) and we arrive at

$$\int_0^x d\beta(y) = -\int_0^x \int_{-\infty}^{y} [cz^3 + Kz]\, d\beta(z)\, dy$$

so that $d\beta$ is absolutely continuous and

$$\beta'(x) = -\int_{-\infty}^{x} [cz^3 + Kz]\, d\beta(z) = -\int_{-\infty}^{x} [cz^3 + Kz]\, \beta'(z)\, dz.$$

Therefore we obtain that β' is absolutely continuous as well, and

$$\beta''(x) = -[cx^3 + Kx]\, \beta'(x).$$

Consequently,

$$\beta'(x) = \text{const} \exp\left[-\frac{c}{4}x^4 - \frac{K}{2}x^2\right] \tag{46}$$

where the constant is uniquely determined by the condition

$$\int_{-\infty}^{\infty} q_0(x)^2\, d\beta(x) = \int_{-\infty}^{\infty} d\beta(x) = 1.$$

We have

$$a_1^2 = \frac{\gamma_0^2(d\beta)}{\gamma_1^2(d\beta)} = \frac{1}{\gamma_1^2(d\beta)} = \int_{-\infty}^{\infty} x^2\, d\beta(x)$$

so that by (46)

$$a_1^2 = \int_{-\infty}^{\infty} x^2 \exp\left[-\frac{c}{4}x^4 - \frac{K}{2}x^2\right] dx \Big/ \int_{-\infty}^{\infty} \exp\left[-\frac{c}{4}x^4 - \frac{K}{2}x^2\right] dx. \tag{47}$$

Thus we proved that if $\{a_n\}$ satisfies (35) and $a_n > 0$ for $n = 1, 2, \ldots$ then a_1 is given by (47), which means that the sequence $\{a_n\}$ is uniquely determined. Thus the theorem has been completely proved.

Acknowledgments

It was the first author [1] who noticed that since the derivatives of the orthogonal polynomials associated with $\exp(-x^k)$, k even, are linear combinations of a bounded number of the same polynomials, one can find pointwise estimates for these polynomials. Applying this observation of the first author, the second author [17] obtained more refined estimates for the orthogonal polynomials corresponding to $\exp(-x^4)$ and these estimates were used to prove asymptotics for such polynomials in [17] and [18]. While the authors presented their ideas at a seminar talk at The Ohio State University, R. Bojanic raised the problem of characterizing all orthogonal polynomials whose derivatives are linear combinations of a bounded number of the same polynomials. This paper grew out of the research initiated by R. Bojanic's problem, and its content was several times discussed with him. The authors express their gratitude to him.

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